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Systems & Control Letters 147 (2021) 104825

Contents lists available at ScienceDirect

Systems & Control Letters

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Well-posedness and exponential stability of boundary control systems with dynamic boundary conditions

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ARTICLE INFO

Article history:
Received 30 October 2019
Received in revised form 22 October 2020
Accepted 29 October 2020
Available online xxxx

Keywords:
Positive semigroup
Feedback theory
Exponential stability
Coupled system
Networks

ABSTRACT

The well-posedness of abstract boundary control systems with dynamic boundary conditions (i.e., boundary conditions evolving according to an operator semigroup acting on the boundary space) is established . Here, the boundary feedback operator is unbounded which makes the investigation more interesting in many applications. The positivity of such problem is well studied. By exploiting the positivity, we find a characterization of exponential stability. Our approach is based on the feedback theory of infinite dimensional linear systems. Finally, these results are applied to coupled systems consisting of a partial differential equation (PDE) and an ordinary differential equation (ODE) and to diffusion processes in networks with dynamic ramification nodes.

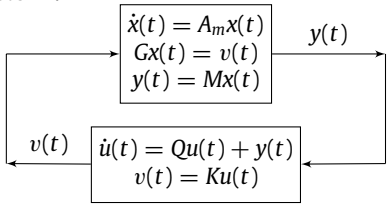
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1. Introduction

There exists an extensive literature on evolution equations (such as heat, Cahn–Hilliard) with dynamically evolving boundary condition, see for instance [1–8]). In particular, the authors of [1,5] worked with the smoothness assumptions on the data, but the representation formula of the solution is not given. In [8], the authors employed an argument based on Lax–Milgram theorem which is not valid in Banach spaces. The aforementioned references were on the dynamic boundary conditions without boundary feedback operators. However, in this paper we study a large class of evolution equations with dynamic boundary conditions on Banach spaces. To define this class, let X, Z and ∂X be Banach spaces such that Z is densely and continuously embedded in X. Let $A_m : D(A_m) = Z \to X$ and $Q : D(Q) \subset \partial X \to \partial X$ be closed differential operators. We also introduce the linear operators $M : Z \to \partial X$ and the boundary operators $G : Z \to \partial X$ and $K : \partial X \to \partial X$. Our approach relies on the theory of feedback perturbations and the theory of infinite dimensional linear systems developed in [9–11]. We look for existence, uniqueness, and asymptotic behavior of solutions of the following equation

$$\begin{cases} \dot{x}(t) = A_m x(t), \quad x(0) = x_0, & t \geq 0, \\ \dot{u}(t) = Qu(t) + Mx(t), \quad u(0) = u_0, & t \geq 0, \\ Gx(t) = Ku(t), & t \geq 0, \end{cases} \tag{1.1}$$

with initial conditions $x_0 \in X$ and $u_0 \in \partial X$. Here, the operator M is called a boundary feedback and the operator K could be a feedback. The problem (1.1) can be presented as the closed loop system :



In the literature (see e.g. [9,12,13]), the commun (minimal) conditions for Eq. (1.1) are $A := A_m$ with $D(A) = \ker(G)$ and $(Q, D(Q))$ are generators of strongly continuous semigroups on X and ∂X, respectively, G is surjective and K is a bounded operator. By using the product state space $\mathcal{X} := X \times \partial X$, the boundary value problem (1.1) can be reformulated as the following abstract Cauchy problem

$$\begin{cases} \dot{w}(t) = \mathcal{A}^M w(t), \quad t \geq 0, \\ w(0) = \binom{x_0}{u_0}, \end{cases} \tag{1.2}$$

where the operator $\mathcal{A}^M$ is defined by

$$\mathcal{A}^M := \begin{pmatrix} A_m & 0 \\ M & Q \end{pmatrix}, \tag{1.3}$$

$$D(\mathcal{A}^M) := \left\{ \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} \in Z \times D(Q) : Gx_0 = Ku_0 \right\}.$$

The main purpose of this paper is two folds, first we introduce the well-posedness of the boundary value problem (1.1) and second to show results on positivity and stability of problem (1.1). To that purpose, we consider the following decomposition

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https://doi.org/10.1016/j.sysconle.2020.104825
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$\mathcal{A}^M := \mathcal{A} + \mathcal{P}$ where

$$\mathcal{A} := \begin{pmatrix} A_m & 0 \\ 0 & Q \end{pmatrix}, \qquad D(\mathcal{A}) := D(\mathcal{A}^M),$$
$$\mathcal{P} := \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} : D(\mathcal{A}) \longrightarrow \mathcal{X}. \tag{1.4}$$

Based on paper [14], we prove that $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup satisfying an appropriate variation of constants formula. Moreover, we introduce conditions for which the operator $\mathcal{P}$ is a Miyadera–Voigt perturbation for $\mathcal{A}$. This will imply that $\mathcal{A}^M$ is a generator of a strongly continuous semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ on $\mathcal{X}$ and thus the perturbed Cauchy problem (1.2) is well-posed. In particular, if the boundary feedback operator M is bounded (i.e $M : X \longrightarrow \partial X$) then $\mathcal{P}$ is a bounded perturbation and hence the generation of $\mathcal{A}^M$ is clearly well obtained. Our approach allows us to easily compute the spectrum of the operator $\mathcal{A}^M$. From this we deduce a necessarily and sufficient condition for which the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ is positive and hence give some information about the exponential stability of this semigroup. The obtained abstract results will be applied to coupled PDE-ODE systems and to diffusion processes in networks with dynamic ramification nodes.

Notation: For a closed linear operator $A : D(A) \subset X \to X$ on a Banach space $(X, \|\cdot\|)$, we denote by $\rho(A)$ the resolvent set of A and by $R(\lambda, A) := (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$, the resolvent operator of A. The spectrum of A is the set $\sigma(A) := \mathbb{C}\backslash\rho(A)$. The spectral bound of A is defined by $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$. On the other hand, using the operator A, we can define a new norm on X by putting

$$\|x\|_{-1} := \|R(\mu, A)x\|, \qquad x \in X, \qquad \mu \in \rho(A).$$

This norm is independent of μ, due to the resolvent equation. The extrapolation of X with respect to A, denoted by X_{-1} is the completion of X with respect to the norm $\|\cdot\|_{-1}$, which is a Banach space. Moreover,

$$(D(A), \|\cdot\|_A) \hookrightarrow (X, \|\cdot\|) \subset (X_{-1}, \|\cdot\|_{-1}),$$

where $\|x\|_A = \|x\| + \|Ax\|$ for $x \in D(A)$ is the graph norm (some times we write $X_{-1,A}$ instead of X_{-1}). If in addition $(A, D(A))$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X, then there exists another C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} which coincides with $(T(t))_{t\geq 0}$ on X and is called the extrapolation semigroup associated to X and A. The generator of this semigroup is the extension of A to X and denoted by $A_{-1} : D(A_{-1}) = X \to X_{-1}$.

2. Well-posedness of dynamic boundary value problems

In this section we study the well-posedness of the dynamic boundary value problem (1.1). According to our discussion in the introductory section, it should be noted that the problem (1.1) is well-posed if and only if the Cauchy problem (1.2) is well-posed if and only if the operator $\mathcal{A}^M$ defined in (1.3) generates a strongly continuous semigroup on $\mathcal{X}$ (for more detail about the well-posedness of Cauchy problem see e.g. [15, Theorem 3.1.12]). To that purpose, we will introduce conditions for which the operators $\mathcal{A}$ and $\mathcal{P}$ defined in (1.4) satisfy the following properties: $(\mathcal{A}, D(\mathcal{A}))$ is a generator on $\mathcal{X}$ and $\mathcal{P}$ is a Miyadera–Voigt perturbation for $\mathcal{A}$. In fact, consider the following standard hypothesis:

(H1) $A := A_m$ with domain $D(A) = \ker G$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X.
(H2) G is surjective.
(H3) Q generates a C_0-semigroup $(S(t))_{t\geq 0}$ on ∂X.

Assumptions **(H1)** and **(H2)** imply (see e.g. [13]) that for each $\lambda \in \rho(A)$, the restriction of G to the kernel $\ker(\lambda - A_m)$ is invertible with inverse denoted by D_λ,

$$D_\lambda : \partial X \longrightarrow \ker(\lambda - A_m) \subset X \tag{2.1}$$

is bounded and called the Dirichlet operator associated to G and A_m. Let us define

$$B := (\lambda - A_{-1})D_\lambda : \partial X \longrightarrow X_{-1}. \tag{2.2}$$

This notation in combination with [13, Lemma 1.3] implies that the operator B is independent of λ. Here, X_{-1} is the extrapolation space associated to X and A and $A_{-1} : X \to X_{-1}$ is the generator of the extrapolation semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} (see notation in Section 1). For some $p \in (1, +\infty)$, we select the operator $\Phi_t : L^p([0, t], \partial X) \to X_{-1}$,

$$\Phi_t v := \int_0^t T_{-1}(t - s)Bv(s)ds, \quad t \geq 0. \tag{2.3}$$

We impose the following important condition

(H4) The operator $B \in \mathcal{L}(\partial X, X_{-1})$ defined in (2.2) satisfies the condition: there exists $t_0 > 0$ such that

$$\mathrm{Range}(\Phi_{t_0}) \subset X.$$

In this case, we say that B is an admissible control operator for A.

According to [16, Section 4.2], the condition **(H4)** implies that for any $t, s \geq 0$ and $u \in L^p([0, s + t], \partial X)$,

$$\begin{aligned} &\Phi_t \in \mathcal{L}(L^p([0, t], \partial X), X), \\ &\Phi_{t+s}u = T(t)\Phi_s u_{|[0,s]} + \Phi_t u(\cdot + s). \end{aligned} \tag{2.4}$$

Moreover for any $0 \leq t \leq \tau$, we have

$$\|\Phi_t\| \leq \|\Phi_\tau\| \tag{2.5}$$

The following proposition introduce the generation of $\mathcal{A}$. Similar results are in [17, Lemma 5.1] and [18, Theorem 3.3].

Proposition 2.1. *Let assumptions* **(H1)** *to* **(H4)** *be satisfied. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ defined in (1.4) generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on $\mathcal{X}$ given by*

$$\mathcal{T}(t) = \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \qquad t \geq 0, \tag{2.6}$$

where

$$R(t)u := \int_0^t T_{-1}(t - s)BKS(s)uds, \quad u \in \partial X.$$

Proof. Let us first show that $(\mathcal{T}(t))_{t\geq 0}$ is a C_0-semigroup on $\mathcal{X}$. Obviously, $\mathcal{T}(0) = I_{\mathcal{X}}$. On the other hand, by using (2.4) we have

$$\begin{aligned} R(t + s)u &= \Phi_{t+s}KS(\cdot)u \\ &= T(t)\Phi_s KS(\cdot)u + \Phi_t KS(\cdot + s) \\ &= T(t)R(s)u + R(t)S(s)u \end{aligned}$$

for any $t, s \geq 0$ and $u \in \partial X$. This implies that $\mathcal{T}(t + s) = \mathcal{T}(t)\mathcal{T}(s)$ on $\mathcal{X}$ for any $t, s \geq 0$. Due to (2.5), for any $t \in (0, 1)$ and $u \in \partial X$, we have

$$\begin{aligned} \|R(t)u\| &= \|\Phi_t KS(\cdot)u\| \\ &\leq \|\Phi_1\|\|K\|\|S(\cdot)u\|_{L^p([0,t],\partial X)} \underset{t\to 0}{\longrightarrow} 0. \end{aligned}$$

This shows that $(\mathcal{T}(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}$. Let $(\tilde{\mathcal{A}}, D(\tilde{\mathcal{A}}))$ be its generator. Clearly for a large $\lambda > 0$ we have

$$R(\lambda, \tilde{\mathcal{A}}) = \int_0^\infty e^{-\lambda t}\mathcal{T}(t)dt$$

$$= \begin{pmatrix} R(\lambda, A) & R(\lambda, A_{-1})BKR(\lambda, Q) \\ 0 & R(\lambda, Q) \end{pmatrix}$$
$$= \begin{pmatrix} R(\lambda, A) & D_\lambda KR(\lambda, Q) \\ 0 & R(\lambda, Q) \end{pmatrix}. \quad (2.7)$$

Let us prove that $(\lambda - \mathcal{A})^{-1}$ exists for large $\lambda > 0$, and $(\lambda - \mathcal{A})^{-1} = R(\lambda, \tilde{\mathcal{A}})$. In fact, let $\binom{x}{u} \in \mathcal{X}$ and $\lambda > 0$ sufficiently large.

Put $x_0 = R(\lambda, A)x + D_\lambda KR(\lambda, Q)u$ and $u_0 = R(\lambda, Q)u$. Then, one deduces that

$$Gx_0 = KR(\lambda, Q)u = Ku_0.$$

By using the fact that $\text{Range}D_\lambda \subset \ker(\lambda - A_m)$, one can see that

$$(\lambda - \mathcal{A})\begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} (\lambda - A_m)x_0 \\ (\lambda - Q)u_0 \end{pmatrix} = \begin{pmatrix} x \\ u \end{pmatrix}.$$

It follows that

$$(\lambda - \mathcal{A})^{-1}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} R(\lambda, A)x + D_\lambda KR(\lambda, Q)u \\ R(\lambda, Q)u \end{pmatrix}$$

By using (2.7), we obtain $R(\lambda, \mathcal{A}) = R(\lambda, \tilde{\mathcal{A}})$, so $\mathcal{A} \equiv \tilde{\mathcal{A}}$. □

Remark 1. Proposition 2.1 can also be proved by using a perturbation result in [14]. In fact, by introducing the operators

$$\mathcal{G} := \begin{pmatrix} G & 0 \end{pmatrix} : Z \times D(Q) \to \partial X,$$
$$\mathcal{M} := \begin{pmatrix} 0 & K \end{pmatrix} \in \mathcal{L}(\mathcal{X}, \partial X).$$

The conditions **(H1)** to **(H4)** show that $\mathcal{G}$ is surjective, the operator

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ 0 & Q \end{pmatrix}, \quad D(\mathcal{A}_0) := \ker \mathcal{G} = D(A) \times D(Q)$$

is a generator of a diagonal semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{X}$, and for $\lambda \in \rho(A) \cap \rho(Q)$, the control operator

$$\mathcal{B} := \begin{pmatrix} B \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda - \mathcal{A}_{0,-1} \end{pmatrix} \mathcal{D}_\lambda$$

is admissible for $\mathcal{A}_0$. Here, $\mathcal{A}_{0,-1} : \mathcal{X} \to \mathcal{X}_{-1}$ is the generator of the extrapolation semigroup $(\mathcal{T}_{0,-1}(t))_{t\geq 0}$ and $\mathcal{D}_\lambda$ is the Dirichlet operator associated to $\mathcal{G}$. As the operator $\mathcal{M}$ is bounded then the triple $(\mathcal{A}_0, \mathcal{B}, \mathcal{M})$ satisfies the conditions of the paper [14, Thm.4.1] and then the operator $(\mathcal{A}, D(\mathcal{A}))$ defined in (1.4) is a generator and coincides with the part of the operator $\mathcal{A}_0^{cl} := \mathcal{A}_{0,-1} + \mathcal{B}\mathcal{M}$ in $\mathcal{X}$. A straightforward calculus shows that

$$\mathcal{A} = \begin{pmatrix} A_{-1} & BK \\ 0 & Q \end{pmatrix},$$
$$D(\mathcal{A}) = \left\{ \binom{x}{u} \in Z \times D(Q) : A_{-1}x + BKu \in X \right\}.$$

In the rest of this section we keep the same conditions **(H1)** to **(H4)** and look for other conditions on M for which the operator $\mathcal{P}$ defined in (1.4) is a Miyadera–Voigt perturbation for $\mathcal{A}$. We first denote by $i : D(A) \to Z$ the continuous injection of the domain $D(A)$ into Z and select

$$C := Mi : D(A) \to \partial X.$$

(H5) For some (hence all) $\alpha > 0$, there exists a constant $\gamma := \gamma(\alpha) > 0$ such that

$$\int_0^\alpha \|CT(s)x\|^p ds \leq \gamma^p \|x\|^p, \quad \forall x \in D(A).$$

We say that C is an admissible observation for A.

For each $\alpha > 0$, we introduce a space

$W_0^{2,p}((0,\alpha), \partial X) := \left\{ f \in W^{2,p}((0,\alpha), \partial X) : f(0) = f'(0) = 0 \right\}$. This space is dense in $L^p([0,\alpha], \partial X)$. By using (2.2), (2.3) and, an integration by part, we obtain

$$\Phi_t f \in Z, \quad \forall f \in W_0^{2,p}((0,\alpha), \partial X), \quad t \geq 0.$$

Hence, we can define the following application

$$(\mathbb{F}f)(t) := M\Phi_t f, \quad t \geq 0, \ f \in W_0^{2,p}((0,\alpha), \partial X). \quad (2.8)$$

This allows us to introduce the following hypothesis:

$(\mathcal{H}_{(A,B,C)})$: The conditions **(H1)** to **(H5)** hold and there exists $\alpha > 0$ such that the operator $\mathbb{F}$ defined by (2.8) extends to a bounded operator $\mathbb{F} \in \mathcal{L}(L^p([0,\alpha], \partial X))$, and the following limit exists for any $\varpi \in \partial X$,

$$\lim_{\tau \to 0^+} \frac{1}{\tau} \int_0^\tau (\mathbb{F}\chi_{[0,\alpha]} \cdot \varpi)(s) ds = 0.$$

Remark 2. In the terminology of infinite dimensional linear systems theory, the condition $(\mathcal{H}_{(A,B,C)})$ means that the triple (A, B, C) generates a regular linear system with state space X, control space ∂X and observation space ∂X with feedthrough operator zero, see e.g. [10,11]. In this case, it is shown in Theorem 5.5 of [11] that $T(t)x \in D(C_\Lambda)$, $\Phi_t f \in D(C_\Lambda)$ and

$$\int_0^\alpha \|C_\Lambda T(s)x\|^p ds \leq \gamma^p \|x\|^p,$$
$$(\mathbb{F}f)(t) = C_\Lambda \Phi_t f$$

for all $x \in X, f \in L^p_{loc}(\mathbb{R}^+, \partial X)$, $\alpha > 0$, and a.e. $t > 0$, where $C_\Lambda : D(C_\Lambda) \subset X \to \partial X$ is the Yosida extension of C with respect to A defined by

$$D(C_\Lambda) = \left\{ x \in X : \lim_{\sigma \to +\infty} C\sigma R(\sigma, A)x \text{ exists in } \partial X \right\}$$
$$C_\Lambda x := \lim_{\sigma \to +\infty} C\sigma R(\sigma, A)x, \qquad x \in D(C_\Lambda).$$

According to [14, Lem.3.6], we have

$$Z \subset D(C_\Lambda) \quad \text{and} \quad (C_\Lambda)_{|Z} = M. \quad (2.9)$$

This result combined with [11] implies that the transfer function $\rho(A) \ni \lambda \longmapsto MD_\lambda$ satisfied

$$\lim_{\lambda \to +\infty} MD_\lambda = 0. \quad (2.10)$$

In addition, there exists $\gamma_0 > \omega_0(A)$ such that

$$\kappa_{\gamma_0} := \sup_{\lambda > \gamma_0} \|CD_\lambda\| < \infty. \quad (2.11)$$

We now state and prove the main result of this section.

Theorem 2.2. *Let the assumption $(\mathcal{H}_{(A,B,C)})$ be verified. Then the following assertions hold:*

(i) The operator $\mathcal{A}^M$ defined by (1.3) generates a C_0-semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ on $\mathcal{X}$.

(ii) For $\lambda \in \rho(A) \cap \rho(Q)$, we have

$$\lambda \in \rho(\mathcal{A}^M) \Longleftrightarrow \lambda \in \rho(Q + MD_\lambda K).$$

(iii) For $\lambda \in \rho(A) \cap \rho(Q) \cap \rho(\mathcal{A}^M)$,

$$R(\lambda, \mathcal{A}^M) = \left(\begin{array}{c|c} R(\lambda, A) + D_\lambda K\mathcal{F}(\lambda)CR(\lambda, A) & D_\lambda K\mathcal{F}(\lambda) \\ \hline \mathcal{F}(\lambda)CR(\lambda, A) & \mathcal{F}(\lambda) \end{array} \right).$$

where $\mathcal{F}(\lambda) := R(\lambda, Q + MD_\lambda K)$.

Proof. First we prove (i). We know that $\mathcal{A}^M = \mathcal{A} + \mathcal{P}$ where the operators $\mathcal{A}$ and $\mathcal{P}$ are given in (1.4). Moreover, Proposition 2.1 given by (2.6) shows that $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ in $\mathcal{X}$. According to [19–21], it suffices to prove that the operator $\mathcal{P} \in \mathcal{L}(D(\mathcal{A}), \mathcal{X})$ is a Miyadera–Voigt perturbation for $\mathcal{A}$. In fact,

let $\binom{x}{u} \in D(\mathcal{A})$ and $\alpha, q > 0$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality, (2.9) and Remark 2, we obtain

$$\begin{aligned}
&\left(\int_0^\alpha \|\mathcal{P}\mathcal{T}(t)\binom{x}{u}\| \, dt\right)^p \\
&\leq \alpha^{\frac{p}{q}} \int_0^\alpha \left\| \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \begin{pmatrix} T(t)x + R(t)u \\ S(t)u \end{pmatrix} \right\|^p dt \\
&= \alpha^{\frac{p}{q}} \int_0^\alpha \|M(T(t)x + R(t)u)\|^p dt \\
&= \alpha^{\frac{p}{q}} \int_0^\alpha \|C_\Lambda(T(t)x + R(t)u)\|^p dt \\
&\leq \alpha^{\frac{p}{q}} 2^{p-1} \int_0^\alpha \|C_\Lambda T(t)x\|^p dt \\
&\qquad + \alpha^{\frac{p}{q}} 2^{p-1} \int_0^\alpha \|C_\Lambda \Phi_t K S(\cdot)u\|^p dt \\
&\leq \alpha^{\frac{p}{q}} 2^{p-1} \gamma(\alpha)^p \|x\|^p \\
&\qquad + \alpha^{\frac{p}{q}} 2^{p-1} \beta(\alpha)^p \int_0^\alpha \|KS(s)u\|^p ds \\
&\leq \left(\alpha^{\frac{1}{q}} \nu(\alpha) \left\|\binom{x}{u}\right\|\right)^p,
\end{aligned}$$

where $\nu(\alpha) > 0$ is a constant depending only on $\alpha > 0, p$ and q with finite limit as $\alpha \to 0$. Now choose $\alpha_0 > 0$ such that $\mu_0 := \alpha_0^{\frac{1}{q}} \nu(\alpha_0) < 1$. Then

$$\int_0^{\alpha_0} \|\mathcal{P}\mathcal{T}(t)\binom{x}{u}\| \, dt \leq \mu_0 \left\|\binom{x}{u}\right\|.$$

This means that $\mathcal{P}$ is a Miyadera–Voigt perturbation for $\mathcal{A}$, and hence $\mathcal{A}^M$ is a generator of a strongly continuous semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ on $\mathcal{X}$.

Let us prove (ii). Let $\lambda \in \rho(\mathcal{A}) = \rho(A) \cap \rho(Q)$, we have

$$\lambda - \mathcal{A}^M = [I - \mathcal{P}R(\lambda, \mathcal{A})](\lambda - \mathcal{A}). \tag{2.12}$$

Then $\lambda \in \rho(\mathcal{A}^M)$ is equivalent to $1 \in \rho(\mathcal{P}R(\lambda, \mathcal{A}))$. According to (1.4) and (2.7), we have

$$I - \mathcal{P}R(\lambda, \mathcal{A}) = \begin{pmatrix} I & 0 \\ -CR(\lambda, A) & I - MD_\lambda KR(\lambda, Q) \end{pmatrix}, \tag{2.13}$$

for all large $\lambda > 0$. Thus

$$\begin{aligned}
\lambda \in \rho(\mathcal{A}^M) &\Longleftrightarrow 1 \in \rho(MD_\lambda KR(\lambda, Q)) \\
&\Longleftrightarrow \lambda \in \rho(Q + MD_\lambda K).
\end{aligned}$$

Let us prove (iii). Let $\lambda \in \rho(A) \cap \rho(Q) \cap \rho(Q + MD_\lambda K)$. First we have

$$R(\lambda, Q + MD_\lambda K) = R(\lambda, Q)(I - MD_\lambda KR(\lambda, Q))^{-1}. \tag{2.14}$$

On the other hand, by (2.12), we have

$$R(\lambda, \mathcal{A}^M) = R(\lambda, \mathcal{A}) \left(I - \mathcal{P}R(\lambda, \mathcal{A})\right)^{-1}. \tag{2.15}$$

Using (2.13) and straightforward computations, we obtain

$$(I - \mathcal{P}R(\lambda, \mathcal{A}))^{-1} = \begin{pmatrix} I & 0 \\ (I - MD_\lambda KR(\lambda, Q))^{-1} CR(\lambda, A) & (I - MD_\lambda KR(\lambda, Q))^{-1} \end{pmatrix} \tag{2.16}$$

Finally, the result follows by combining (2.7), (2.13), (2.14), (2.15) and (2.16). □

We end this section by giving the explicit expression of the solution of the dynamic boundary equation (1.1).

Proposition 2.3. *Let the assumption $(\mathcal{H}_{(A,B,C)})$ be verified. For any initial condition $w(0) := \binom{x_0}{u_0} \in \mathcal{X}$, the dynamic boundary equation (1.1) admits a unique solution*

$$w : [0, \infty) \to \mathcal{X}, \qquad w(t) = \binom{x(t)}{u(t)}$$

such that

$$\begin{aligned}
&x(t) \in D(C_\Lambda), \quad a.e.\ t > 0, \\
&x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)BKu(s)ds, \quad t \geq 0, \text{ and} \\
&u(t) = S(t)u_0 + \int_0^t S(t-s)C_\Lambda x(s)ds, \quad t \geq 0.
\end{aligned}$$

Proof. The well-posedness of (1.1) is given in Theorem 2.2. The solution of (1.1) is given by

$$w(t) = \mathcal{T}^M(t)\binom{x_0}{u_0}, \quad t \geq 0.$$

On the other hand, in the proof of this theorem we have seen that $\mathcal{P}$ is in fact an admissible observation operator for $\mathcal{A}$. Now, according to [21, Thm.2.1], we have

$$\begin{aligned}
&w(t) \in D(\mathcal{P}_\Lambda), \quad a.e.\ t > 0, \\
&w(t) = \mathcal{T}(t)\binom{x_0}{u_0} + \int_0^t \mathcal{T}(t-s)\mathcal{P}_\Lambda w(s)ds, \quad t \geq 0,
\end{aligned} \tag{2.17}$$

where $(\mathcal{T}(t))_{t\geq 0}$ is the C_0-semigroup generated by $\mathcal{A}$ and $\mathcal{P}_\Lambda$ is the Yosida extension of $\mathcal{P}$ with respect to $\mathcal{A}$. Let us compute this extension. Let λ be sufficiently large. Using (1.4) and (2.7), we have

$$\mathcal{P}\lambda R(\lambda, \mathcal{A}) = \begin{pmatrix} 0 & 0 \\ C\lambda R(\lambda, A) & MD_\lambda K\lambda R(\lambda, Q) \end{pmatrix} \tag{2.18}$$

By using (2.11) and (2.10), for any $\overline{u} \in \partial X$, we have

$$\begin{aligned}
\|MD_\lambda K\lambda R(\lambda, Q)\overline{u}\| &\leq \|MD_\lambda K(\lambda R(\lambda, Q)\overline{u} - \overline{u})\| \\
&\qquad + \|MD_\lambda K\overline{u}\| \\
&\leq \kappa_{\gamma_0}\|K\|\|\lambda R(\lambda, Q)\overline{u} - \overline{u}\| \\
&\qquad + \|MD_\lambda\|\|K\overline{u}\|.
\end{aligned}$$

This shows that $MD_\lambda K\lambda R(\lambda, Q)\overline{u}$ goes to zero as $\lambda \to +\infty$. Thus, by (2.18), we have

$$D(\mathcal{P}_\Lambda) = D(C_\Lambda) \times \partial X, \quad \mathcal{P}_\Lambda = \begin{pmatrix} 0 & 0 \\ C_\Lambda & 0 \end{pmatrix}. \tag{2.19}$$

By combining (2.6), (2.17) and (2.19)

$$w(t) = \binom{x(t)}{u(t)} \in D(C_\Lambda) \times \partial X, \qquad a.e.\ t > 0$$

and for any $\binom{x_0}{u_0}$ and $t \geq 0$,

$$\binom{x(t)}{u(t)} = \begin{pmatrix} T(t)x_0 + R(t)u_0 + \int_0^t R(t-s)C_\Lambda x(s)ds \\ S(t)u_0 + \int_0^t S(t-s)C_\Lambda x(s)ds \end{pmatrix}$$

By using a change of variables and Fubini theorem, we obtain

$$\begin{aligned}
x(t) &= T(t)x_0 + \int_0^t T_{-1}(t-s)BKS(s)u_0 ds \\
&\quad + \int_0^t \int_0^{t-s} T_{-1}(t-s-\sigma)BKS(\sigma)C_\Lambda x(s)d\sigma ds \\
&= T(t)x_0 + \int_0^t T_{-1}(t-s)BKS(s)u_0 ds \\
&\quad + \int_0^t \int_s^t T_{-1}(t-\tau)BKS(\tau-s)C_\Lambda x(s)d\tau ds \\
&= T(t)x_0 + \int_0^t T_{-1}(t-s)BKS(s)u_0 ds \\
&\quad + \int_0^t T_{-1}(t-\tau)BK\int_0^\tau S(\tau-s)C_\Lambda x(s)ds d\tau \\
&= T(t)x_0 + \int_0^t T_{-1}(t-\tau)BKu(\tau)dt. \quad \square
\end{aligned}$$

3. Exponential stability of dynamic boundary value problems

In the previous section, we saw that under the condition $(\mathcal{H}_{(A,B,C)})$ the boundary dynamic equation (1.1) is solved by a strongly continuous semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ on $\mathcal{X}$, see Theorem 2.2. In this section, we will prove some results on the stability of this equation.

We will first recall certain definitions and certain results on positivity and stability. Let $(V(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space E with generator $\Upsilon : D(\Upsilon) \subset E \to E$.

Definition 3.1. The semigroup $(V(t))_{t\geq 0}$ is called:

(i) uniformly exponentially stable if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\|V(t)\| = 0.$$

(ii) exponentially stable if there exists $\varepsilon > 0$ such that

$$\lim_{t\to\infty} e^{\varepsilon t}\|V(t)x\| = 0, \qquad \forall x \in D(\Upsilon).$$

We mention that the uniform exponential stability of the semigroup $(V(t))_{t\geq 0}$ is equivalent to the fact that the type $\omega_0(V) < 0$. Clearly, uniform exponential stable semigroups are exponential stable. However, the converse is not true even the semigroup is positive, where it is justified by an example (see [15, Example 5.1.11]). In the following, we will give a result on exponential stability of the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ based on the concept of positivity. Before doing so, we need to recall results from positive semigroups. We then assume that E is a Banach lattice (see e.g. [22,23]). The semigroup $(V(t))_{t\geq 0}$ is called positive if each operator $T(t)$ is positive, that is, if $0 \leq f \in E$ implies $0 \leq V(t)f$ for each $t \geq 0$. We recall that $(V(t))_{t\geq 0}$ is positive if and only if the resolvent $R(\lambda, \Upsilon)$ of its generator Υ is positive for all sufficiently large λ, see e.g. [22, p. 168]. On the other hand, if $(V(t))_{t\geq 0}$ is positive and the spectral bound satisfies $s(\Upsilon) > -\infty$, then $s(\Upsilon) \in \sigma(\Upsilon)$. In addition, for $\lambda_0 \in \rho(\Upsilon)$, $R(\lambda_0, \Upsilon)$ is positive if and only if $\lambda_0 > s(\Upsilon)$ (see e.g. [22, p. 188]).

According to [15, Theorem 5.3.1 and Exemple 5.3.2], we have the following result: A positive semigroup $(V(t))_{t\geq 0}$ on a Banach Lattice E with generator $\Upsilon : D(\Upsilon) \subset E \to E$ is exponentially stable (i.e. $e^{\varepsilon t}\|V(t)f\| \to$ as $t \to \infty$ for all $f \in D(\Upsilon)$ and for some $\varepsilon > 0$) if and only if $s(\Upsilon) < 0$.

We need the following hypothesis.

Assumption 1. We assume that the condition $(\mathcal{H}_{(A,B,C)})$ is satisfied. Moreover, we suppose that

- X and ∂X are Banach lattices,
- A and Q generate positive semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on X and ∂X, respectively,
- M and K are positive operators,
- The Dirichlet operator D_λ is positive for λ sufficiently large.

Remark 3. Assumption 1 implies that D_λ is decreasing and positive for all $\lambda > s(A)$. In fact, let λ sufficiently large. The resolvent equation and (2.2) show that for any $\lambda \geq \mu > s(A)$,

$$D_\mu - D_\lambda = (\lambda - \mu)R(\mu, A)D_\lambda \geq 0.$$

Theorem 3.2. *Under Assumption 1, the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ obtained in Theorem 2.2 is positive in the Banach lattice $X \times \partial X$.*

Proof. Let $(\mathcal{T}^M(t))_{t\geq 0}$ be the semigroup given in Theorem 2.2. To prove the positivity of the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$, it suffices to show that $R(\lambda, \mathcal{A}^M)$ is positive for a large $\lambda > 0$. According to Theorem 2.2, it suffices to prove that $R(\lambda, Q + MD_\lambda K)$ is positive. In fact, by virtue of (2.11), there exist constants $\gamma_0 > 0$ and $\kappa > 0$ such that

$$\|MD_\lambda\| \leq \kappa \text{ for all } \lambda > \gamma_0.$$

Let $\lambda > \max(\omega, \gamma_0)$ with $\omega > \omega_0(Q)$. Using the resolvent estimate of the generator Q, there exists a constant $b > 0$ such that

$$\|MD_\lambda KR(\lambda, Q)\| \leq \kappa\|K\|\frac{b}{\lambda - \omega}.$$

Now, for $\lambda > \max(\gamma_0, \omega + 2\kappa\|K\|b) := \gamma_1$,

$$\|MD_\lambda KR(\lambda, Q)\| \leq \frac{1}{2}.$$

Hence, for such $\lambda > \gamma_1$, the resolvent $R(\lambda, Q + CD_\lambda K)$ is given by the Neumann series

$$R(\lambda, Q + MD_\lambda K) = R(\lambda, Q)\sum_{n\geq 0}(MD_\lambda KR(\lambda, Q))^n.$$

As the operators M, K, D_λ and $R(\lambda, Q)$ are positive then the resolvent $R(\lambda, Q + MD_\lambda K)$ is positive for $\lambda > \gamma_1$. □

In the following result, we introduce conditions for the exponential stability of the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ in the sense that $\|e^{\varepsilon t}\mathcal{T}^M(t)f\| \to 0$, $t \to \infty$ for all $f \in D(\mathcal{A}^M)$ and some $\epsilon > 0$ (see our discussion at the beginning of this section)

Theorem 3.3. *Let Assumption 1 be satisfied. Assume that A generates an exponentially stable semigroup. Then the semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ is exponentially stable if and only if $s(Q + MD_0K) < 0$.*

Proof. According to Theorem 3.2, the C_0-semigroup $(\mathcal{T}^M(t))_{t\geq 0}$ is positive. Then the exponential stability of this semigroup is characterized by $s(\mathcal{A}^M) < 0$.

Let us now show that

$$s(\mathcal{A}^M) < 0 \Longleftrightarrow s(Q + MD_0K) < 0.$$

In fact, suppose that $s(\mathcal{A}^M) < 0$. By using the positivity of semigroup $(\mathcal{T}^M(t))_{t\geq 0}$, we have

$$0 \in \rho(\mathcal{A}^M) \text{ and } R(0, \mathcal{A}^M) \geq 0.$$

Let $\begin{pmatrix}0\\ f\end{pmatrix}$ be a positive element in the Banach lattice $X \times \partial X$. Then by Theorem 2.2, we have

$$R(0, \mathcal{A}^M)\begin{pmatrix}0\\ f\end{pmatrix} = \begin{pmatrix}D_0KR(0, Q + MD_0K)f\\ R(0, Q + MD_0K)f\end{pmatrix} \geq 0.$$

According to Remark 3, the Dirichlet operator D_0 is positive. It follows that

$$R(0, Q + MD_0K) \geq 0.$$

From the proof of Theorem 3.2, the semigroup generated by $Q + MD_0K$ is positive. Therefore,

$$s(Q + MD_0K) < 0.$$

Conversely, we suppose that $s(Q + MD_0K) < 0$. Using the positivity of semigroup generated by $Q + MD_0K$,

$$0 \in \rho(Q + MD_0K) \text{ and } R(0, Q + MD_0K) \geq 0.$$

By Theorem 2.2, we conclude that

$$0 \in \rho(\mathcal{A}^M) \text{ and } R(0, \mathcal{A}^M) \geq 0.$$

This suggests that $s(\mathcal{A}^M) < 0$. □

4. An application to a coupled system PDE-ODE

Consider a cascade system of a finite dimensional system and an infinite dimensional system

$$\begin{cases} \dot{v}(t,\cdot) = \sigma A_m v(t,\cdot), & t \geq 0; \\ \dfrac{dZ(t)}{dt} = aZ(t) + bLv(t,\cdot), & t \geq 0; \\ Gv(t,\cdot) = V(t), \end{cases} \tag{4.1}$$

where $A_m : D(A_m) \subset X \longrightarrow X$ is a differential operator in a Banach space $X := L^p(\Omega)$ (with $\Omega \subset \mathbb{R}^n$), the boundary operator $G : D(A_m) \longrightarrow \mathbb{R}$ is a surjective, the operator $L : D(A_m) \longrightarrow \mathbb{R}$ is interconnection between ODE and PDE, $a \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times 1}$, $\sigma \in (0,\infty)$, and $V(t) : \mathbb{R}^+$ is an input function. Here, $Z(t)$ is the solution of an *ODE* while $v(t)$ is a solution of a PDE. We assume that the initial condition $(Z(0), v(0,\cdot))$ belongs to the state space $\mathcal{X} := \mathbb{R}^n \times X$.

In the case of $\sigma = 1$ and particular PDE such as heat, wave or translation equations with Dirichlet and Neumann conditions, the stability and stabilization of (4.1) were investigated in e.g [24–28]. The approach used in the aforementioned references is mainly based on the backstepping method.

We now consider a feedback law

$$\text{“}V(t) = kZ(t)\text{”}, \tag{4.2}$$

where $k \in \mathbb{R}^{1\times n}$. By combining (4.1) and (4.2), we obtain the coupled system

$$\begin{cases} \dot{v}(t,\cdot) = \sigma A_m v(t,\cdot), & t \geq 0; \\ \dfrac{dZ(t)}{dt} = aZ(t) + bLv(t,\cdot), & t \geq 0; \\ Gv(t,\cdot) = kZ(t). \end{cases} \tag{4.3}$$

Recently, the exponential stability of coupled system (4.3) has been established for heat-ODE or transport-ODE e.g. see [29,30] which are based on Lyapunov functional technique and some integral inequality. We mention that there exist other works dealing with well-posedness, controllability and observability of a class of coupled systems, see e.g. [31–33].

In this section we will establish the exponential stability of system (4.3) via the positivity consideration. In order to use our results (see Section 3), we assume that the semigroup $(e^{ta})_{t\geq 0}$ and the vectors b, k and operator L are all positive. We further assume that

$$A := A_m, \quad D(A) = \ker G,$$

generates an exponentially stable and a positive semigroup $(T(t))_{t\geq 0}$ on X.

The operators D_λ and B are defined as in (2.1), (2.2). We select

$$L_0 = L\imath : D(A) \longrightarrow \mathbb{R},$$

where $\imath$ is the injection from $D(A_m)$ into $D(A)$.

The following lemma can be obtained by a straightforward calculus.

Lemma 4.1. *The following assertions hold:*

- *The operator*

  $$\sigma A := \sigma A_m, \quad D(\sigma A) = D(A)$$

  generates an exponentially stable and a positive semigroup $(T(\sigma t))_{t\geq 0}$.
- $\lambda \in \rho(A) \Longleftrightarrow \sigma\lambda \in \rho(\sigma A)$ *and*

  $$R(\lambda, A) = \sigma R(\sigma\lambda, \sigma A).$$

- *The Dirichlet operator* D^σ_λ *associated to* σA_m, G *is given by*

  $$D^\sigma_\lambda = D_{\frac{\lambda}{\sigma}} \quad \text{for } \lambda \in \rho(\sigma A). \tag{4.4}$$

- *The condition* $(\mathcal{H}_{(A,B,L_0)})$ *is satisfied if and only if the condition* $(\mathcal{H}_{(\sigma A,\sigma B,L_0)})$ *is satisfied.*

Theorem 4.2. *Let assumption* $(\mathcal{H}_{(A,B,L_0)})$ *be satisfied. If the Dirichlet operator* D_λ *is positive for* λ *sufficiently large then the coupled system* (4.3) *admits a unique positive solution.*

Proof. The proof immediately follows by combining Lemma 4.1 , Theorems 2.2 and 3.2. □

Theorem 4.3. *Let assumption* $(\mathcal{H}_{(A,B,L_0)})$ *be satisfied. If the Dirichlet operator* D_λ *is positive for* λ *sufficiently large then the coupled system* (4.3) *is exponentially stable if and only if* $s(a + bLD_0k) < 0$ *(i.e:* $a + bLD_0k$ *is Hurwitz).*

Proof. The proof is a direct consequence of Lemma 4.1 and Theorem 3.3. □

Example 4.4. Consider the coupling system of finite dimensional system and a heat equation via Dirichlet interconnection which is described as the following

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \gamma \dfrac{\partial^2 v}{\partial x^2}(x,t), & t \geq 0, x \in (0,\pi); \\ \dfrac{dZ(t)}{dt} + Z(t) = 2v(0,t), & t \geq 0; \\ \dfrac{dv}{dx}(0,t) + kZ(t) = 0, & t \geq 0; \\ v(\pi,t) = 0, & t \geq 0 \end{cases} \tag{4.5}$$

where $Z(t) \in \mathbb{R}$ is the state of the ODE and $v(t,\cdot) \in L^2[0,\pi]$ is the state of the PDE with the initial condition $(Z(0), v(0,\cdot)) \in \mathbb{R} \times L^2[0,\pi]$. The constant $\gamma \in (0,+\infty)$ is a thermal diffusivity . We set

$$A_m := \frac{d^2}{dx^2}, \quad D(A_m) = \{f \in H^2(0,\pi) : f(\pi) = 0\}.$$

It is well-known that the operator $A := A_m$ with domain $D(A) := \{f \in D(A_m) : f'(0) = 0\}$ generates a positive and an exponentially stable semigroup (cf. [16, Exemple 2.6.8]). By standard computations, one can see that the Dirichlet operator associated to A_m and $-\delta'_0$ is given by

$$(D_\lambda u)(x) = \begin{cases} \dfrac{\sinh(\sqrt{\lambda}(\pi - x))}{\sqrt{\lambda}\cosh(\sqrt{\lambda}\pi)}u, & \lambda > 0, x \in [0,\pi]; \\ (\pi - x)u, & \lambda = 0 \end{cases}$$

for all $u \in \mathbb{R}$. Hence, from (4.4), the Dirichlet operator associated to γA_m and $-\delta'_0$ is

$$(D^\gamma_\lambda u)(x) = \begin{cases} \dfrac{\sinh(\sqrt{\frac{\lambda}{\gamma}}(\pi - x))}{\sqrt{\frac{\lambda}{\gamma}}\cosh(\sqrt{\frac{\lambda}{\gamma}}\pi)}u, & \lambda > 0, x \in [0,\pi]; \\ (\pi - x)u, & \lambda = 0 \end{cases}$$

According to [34, Example 1], the assumption $(\mathcal{H}_{(A,-A_{-1}D_0,\delta_0)})$ is verified. It follows from Theorems 4.2 and 4.3 that the coupled heat-ODE system (4.5) admits a unique positive solution, and it is exponentially stable if and only if $0 \leq k < \frac{1}{2\pi}$.

Example 4.5. Consider the coupling system of an finite dimensional system and a transport equation as follows

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = \rho \dfrac{\partial v}{\partial x}(x,t), & t \geq 0, x \in (0,1); \\ \dfrac{dZ(t)}{dt} + 2Z(t) = v(0,t), & t \geq 0; \\ v(1,t) = kZ(t), & t \geq 0 \end{cases} \tag{4.6}$$

where $Z(t) \in \mathbb{R}$ is the state of the ODE and $v(t,\cdot) \in L^p[0,1]$ is the state of the PDE of transport with the initial condition $(Z(0), v(0,\cdot)) \in \mathbb{R} \times L^p[0,1]$. The constant $\rho \in (0,\infty)$ is the propagation velocity and $k \in [0,+\infty)$. Let

$$A_m := \frac{d}{dx}, \quad D(A_m) = W^{1,p}(0,1).$$

It is well-known that the operator $A := A_m$ with domain $D(A) := \{f \in D(A_m) : f(1) = 0\}$ generates a nilpotent translation semigroup on $L^p[0,1]$. Hence, the semigroup is positive and exponentially stable. Clearly, the Dirichlet operator associated to ρA_m and δ_1 is

$$(D^\rho_\lambda u)(x) = e^{\frac{\lambda}{\rho}(x-1)}u, \qquad x \in [0,1], \quad u \in \mathbb{R}.$$

Due to [10, Example 2.3.4], the assumption $(\mathcal{H}_{(A,-A_{-1}D_0,\delta_0)})$ is verified. From Theorems 4.2 and 4.3, the coupled transport-ODE (4.6) admits a unique positive solution, and it is exponentially stable if and only if $k \in [0,2)$.

5. Diffusion processes in networks with dynamic ramification nodes

In the past two decays, the flows in networks have been taken a lot of interest thanks to their successful applications (e.g. neural networks, chemical processes, population biology, ...). Moreover, the transport and diffusion process in networks with static ramification nodes have been studied quite intensively in the literature, see e.g. [35–43], while the networks with dynamic ramification nodes have been established in [44] for transport processes. The aim of this section is to study the networks with dynamic ramification nodes for diffusion processes. For that, let us first introduce some elementary notions of graph theory.

Definition 5.1. Let $V := \{v_1, \ldots, v_n\}$ and $E := \{e_1, \ldots, e_m\}$ be two disjoint sets and let $G : E \longrightarrow V \times V$ be a function.

- The triplet (V, E, G) is called a direct graph (for simplicity it is denoted by G).
- The elements of V and E are called respectively the vertices and the edges of the graph.
- We say that the edge e_j connects the vertices v_i and v_p if $G(e_j) = (v_i, v_p)$. Moreover, v_i is called the tail of e_j and v_p is called the head of e_j.
- The edge e_j is called a loop if its tail coincides with its head.

Definition 5.2. A directed graph G is called simple if it contains no loops and no multiple edges.

Definition 5.3.

- We define the outgoing incidence matrix $\Phi^- = (\phi^-_{ij})_{n\times m}$ of G by
$$\phi^-_{ij} := \begin{cases} 1, & \text{if } v_i \text{ is the tail of } e_j, \\ 0, & \text{otherwise.} \end{cases}$$
- The matrix $\Phi^-_w = (w^-_{ij})_{n\times m}$ with entries $w^-_{ij} \geq 0$ is said to be the weighted outgoing incidence matrix of G if satisfies
$$w^-_{ij} = 0 \Longleftrightarrow \phi^-_{ij} = 0.$$

Definition 5.4.

- We define the incoming incidence matrix $\Phi^+ = (\phi^+_{ij})_{n\times m}$ of G by
$$\phi^+_{ij} := \begin{cases} 1, & \text{if } v_i \text{ is the head of } e_j, \\ 0, & \text{otherwise.} \end{cases}$$
- The matrix $\Phi^+_w = (w^+_{ij})_{n\times m}$ with entries $w^+_{ij} \geq 0$ is said to be the weighted incoming incidence matrix of G if satisfies
$$w^+_{ij} = 0 \Longleftrightarrow \phi^+_{ij} = 0.$$

Inspired from the work of Sikolya [44], we introduce the following diffusion process on simple graph G which the edges are parameterized by the interval $[0,1]$, in the opposite direction of the flow, with dynamic at boundary :

$$\begin{cases} \dfrac{\partial}{\partial t}u_j(t,x) = c_j \dfrac{\partial^2}{\partial x^2}u_j(t,x) - b_j u_j(t,x), \\ \qquad\qquad\qquad\qquad x \in (0,1), t > 0, \\ \dfrac{d}{dx}u(t,1) \in ran(\Phi^-_w)^\top, \qquad t \geq 0, \\ \dfrac{d}{dx}u(t,0) = 0, \qquad t \geq 0, \\ \dfrac{d}{dt}\left(\Phi^- \dfrac{d}{dx}u(t,1)\right) = k\Phi^+_w u(t,0) + \mathbb{Q}\Phi^- \dfrac{d}{dx}u(t,1), \\ \Phi^- \dfrac{d}{dx}u(0,1) = y_0 \in \mathbb{C}^n, \quad u_j(0,\cdot) = f_j \in L^2(0,1), \end{cases} \tag{5.1}$$

where $u(t,x) = (u_1(t,x), u_2(t,x), \ldots, u_m(t,x))$ for $x \in [0,1]$, $t \geq 0$ and the numbers c_j, b_j and k are strictly positive for $j = 1,2,\ldots,m$ and $\mathbb{Q} \in \mathbb{C}^{n\times n}$, here c_j present the diffusivity of the flow on each edge e_j.

To the best of our knowledge, the well-posedness and asymptotic behavior of problem (5.1) is not yet studied. Our interest here is to apply the previous results in order to show the well-posedness and exponential stability of problem (5.1). To that purpose, we introduce the spaces

- $X := (L^2[0,1])^m \cong L^2([0,1],\mathbb{C}^m)$, the space of functions on the edges.
- $\partial X := \mathbb{C}^n$, the (boundary) space of the values in the vertices.
- $Z := \{f \in H^2([0,1],\mathbb{C}^m) : f'(0) = 0, f'(1) \in ran(\Phi^-_w)^\top\}$, the solution space.

and operators

- $A_m : Z \longrightarrow X$ is the differential (maximal) operator defined by
$$A_m := \begin{pmatrix} c_1\frac{d^2}{dx^2} - b_1 & & 0 \\ & \ddots & \\ 0 & & c_m\frac{d^2}{dx^2} - b_m \end{pmatrix}.$$
- $G : Z \longrightarrow \partial X$ is the trace (boundary) operator defined by $G := \Phi^- \otimes \delta'_1$.
- $M : Z \longrightarrow \partial X$ is the feedback boundary operator defined by $M := \Phi^+_w \otimes \delta_0$.

Now let us define

$$A := A_m, \qquad D(A) := \{f \in Z : f'(1) = 0\}.$$

Hence, A generates a positive and exponentially stable semigroup $(T(t))_{t\geq 0}$.

According to [44, Definition 4.2 and remark 4.3], let Φ^-_w be the matrix as in Definition 5.3 and define ω_i the ith row sum of

matrix Φ_w^- for $i = 1, \ldots, n$. Let $\widetilde{\Phi}_w^- := (\widetilde{w}_{ij})_{n\times m}$ be the weighted outgoing incidence matrix with entries

$$\widetilde{w}_{ij} := \frac{w_{ij}}{\omega_i}, \quad i = 1, \ldots, n \quad j = 1, \ldots, m.$$

In addition, $\widetilde{\Phi}_w^-$ is called row stochastic and satisfied

$$\Phi^-(\widetilde{\Phi}_w^-)^\top = \mathbb{1},$$

where $\mathbb{1}$ is the $n \times n$ identity matrix. Therefore, the Dirichlet operator associated to A_m and G is

$$(D_\lambda u)(x) := \begin{pmatrix} \alpha_\lambda^1(x) & & 0 \\ & \ddots & \\ 0 & & \alpha_\lambda^m(x) \end{pmatrix} \cdot ((\widetilde{\Phi}_w^-)^\top u) \tag{5.2}$$

where

$$\alpha_\lambda^j(x) := \frac{\cosh\left(\sqrt{\frac{\lambda+b_j}{c_j}}x\right)}{\sqrt{\frac{\lambda+b_j}{c_j}}\sinh\left(\sqrt{\frac{\lambda+b_j}{c_j}}\right)},$$

for $u \in \partial X$, $x \in [0, 1]$ and $j = 1, \ldots, m$. Furthermore, the transfer function associated to A_m, G and M is

$$H_\lambda := MD_\lambda = \Phi_\omega^+ \begin{pmatrix} \alpha_\lambda^1(0) & & 0 \\ & \ddots & \\ 0 & & \alpha_\lambda^m(0) \end{pmatrix} (\widetilde{\Phi}_w^-)^\top,$$

which is a matrix with entries

$$(H_\lambda)_{ip} = \begin{cases} w_{ij}^+ \frac{2}{\sqrt{\frac{\lambda+b_j}{c_j}}\sinh\left(\sqrt{\frac{\lambda+b_j}{c_j}}\right)} \widetilde{w}_{pj}^-, \\ \qquad \text{if } v_i = e_j(0) \text{ and } v_p = e_j(1), \\ 0, \qquad \text{otherwise.} \end{cases}$$

Theorem 5.5. *The problem (5.1) is well-posed, that is, the operator*

$$A_D := \begin{pmatrix} A_m & 0 \\ M & \mathbb{Q} \end{pmatrix},$$

$$D(A_D) := \left\{ \begin{pmatrix} f \\ y \end{pmatrix} \in Z \times \mathbb{C}^n : Gf = ky \right\}$$

generates a C_0-semigroup $(T_D(t))_{t\geq 0}$ on $X \times \partial X$.

Proof. According to Theorem 2.2, it is enough to show that $(A, -A_{-1}D_0, M\imath)$ generates a regular linear system where $\imath : D(A) \mapsto Z$. For the sake of simplicity, we consider $m = n = 1$. Let us define the operator $A_0 f = f''$ with domain $D(A_0) = D(A)$. Hence, A_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on $L^2[0, 1]$ and is a self-adjoint. Now let $f \in Z$, and $g \in D(A_0^*) = D(A_0)$. Thus, using integration by parts, we have

$$\langle f'', g\rangle - \langle f, A_0^*\rangle = f'(1)\overline{g(1)}.$$

Due to [16, Remark 10.1.6], we have $(-(A_0)_{-1}D_0)^* = \delta_1$. Since δ_1 is an admissible observation operator for A_0 then by duality, $-(A_0)_{-1}D_0$ is an admissible control operator for A_0. On the other hand, δ_0 is an admissible observation for A_0. Moreover, the transfer function associated to $\frac{d^2}{dx^2}$, $-(A_0)_{-1}D_0$ and δ_0 is bounded on some right half-plan. This implies that the triple $(A_0, -(A_0)_{-1}D_0, \delta_0\imath)$ generates a regular linear system (see e.g. [17, Proposition 2.1]). According to Lemma 4.1, the triple $(cA_0, -(cA_0)_{-1}D_0, \delta_0\imath)$ generates a regular linear system and hence $(cA_0 + b, -(cA_0 + b)_{-1}D_0, \delta_0\imath)$ generates a regular linear system as well for $c, b > 0$. This means that the triple $(A, -A_{-1}D_0, M\imath)$ generates a regular linear system. □

Theorem 5.6. *If $\mathbb{Q}$ is real and positive off-diagonal then the semigroup $(T_D(t))_{t\geq 0}$ generated by A_D is positive.*

Proof. We shall use Theorem 3.2. For this we check Assumption 1. Since the matrix $\mathbb{Q}$ is real and positive off-diagonal and the dimension of ∂X is finite then the semigroup $(e^{\mathbb{Q}t})_{t\geq 0}$ generated by matrix $\mathbb{Q}$ is positive. Because Φ_w^+ is a positive matrix, the operator M is also positive. Furthermore, one can see from (5.2) that the Dirichlet operator D_λ is positive for large $\lambda > 0$. Finally, the rest of proof follows from Theorems 3.2 and 5.5. □

Proposition 5.7. *If $\mathbb{Q}$ is real and positive off-diagonal then the semigroup $(T_D(t))_{t\geq 0}$ is exponentially stable if and only if the matrix $\mathbb{Q} + kH_0$ is Hurwitz.*

Proof. The proof follows immediately from Theorems 5.6 and 3.3. □

Remark 4. It should be noted that the semigroup $(T(t))_{t\geq 0}$ generated by A is immediately compact and $R(\lambda, \mathbb{Q})$ has finite dimensional range hence by perturbation theory (see [34]), the semigroup $(T_D(t))_{t\geq 0}$ generated by A_D is immediately compact. This implies that $w_0(A_D) = s(A_D)$ which means that, under the assumption of Proposition 5.7, the semigroup $(T_D(t))_{t\geq 0}$ is uniformly exponentially stable if and only if $\mathbb{Q} + kH_0$ is Hurwitz.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors wish to thank the Editor and the referees for thoughtful comments and constructive suggestions.

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